Logic and Medicine
On *Institutio logica* by Galen

Michele Malatesta

1. *Institutio logica* by Galen

*Institutio logica* by Galen, discovered toward the middle of the 19th century by Minoides Mynas in one of the monasteries of Mount Athos, has not yet obtained the place that it deserves in the history of logic and of medicine. Even if you did not know that the book is a work by the Pergamus’ physician, you could deduce by means of the inner analysis of the text that its author is a medical scientist. In fact not only the continuous instances, drawn mostly from physiology, belong to the medical sciences, but also the demonstrative process clearly proves its use in medicine in general and in the diagnostic field particularly.

Galen has several merits: his book overcomes the conflict between the Peripatetic logicians and the Stoic ones, showing a reconcilement of opposed standpoints and anticipating, in this way, Freges’ *Begriffsschrift* (1878) perspective. Moreover, it contains a complete analysis of three different senses of the connective “either... or...” and studies not only dyadic but also *n*-adic (*n*≥3) logical functions.

2. Stoic logic

For Stoics philosophy is divided in three parts (*trimer*): logic, physics and ethics. Logic, in turn, is divided in two sections: dialectic and rhetoric. In the dialectic range Stoics

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1 DIOG. LAERT. *Vita*, VII, 39.
2 DIOG. LAERT. *Vita*, VII, 41. “But Stoic dialectics covered a wider field than our formal logic – it included also theory of knowledge, with an ample
distinguish two sorts of propositions (§xiîmata): the simple (•pl≠) and the non-simple ones (oεc •pl≠). The modern world speaks of atomic and molecular sentences respectively.

«For the Dialecticians proclaim that almost the first and most important distinction in propositions is that by which some of them are simple, others not simple [...] propositions are called simple since they are not compounded of propositions but of certain other things. For example ‘it is day’ is a simple proposition inasmuch as it is neither formed from the same proposition twice repeated nor compounded of different propositions, but is constructed of certain other elements, namely ‘day’ and ‘it is’. Moreover, there is no connective in it either. And not simple are those which are, so to say, double, and all such as are compounded of a proposition twice repeated, or of different propositions, by means of one or more connectives, as for example ‘if it is day, it is day’; ‘if it is night, it is dark’; ‘both day exists and light exists’; ‘either day exists or night exists’ (toi òmûra καταΚ ουStan & ηx Kat Stan).» Sext. Emp. Adv. Math. VIII, 93-95.

Stoics include among non-simple propositions the conditional (συνημμûnon), the disjunction (διεζευγμûnon) and the paradisjunction (paradiezeugmûnon). The first, also called Philonian implication, corresponds to the modern material implication, the second to the modern exclusive disjunction or non-equivalence and the third to the modern inclusive disjunction or logical sum. See the following table for a comparison of the first logical function in both perspectives:

<table>
<thead>
<tr>
<th>STOICS</th>
<th>MODERN LOGIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(sunhmmûnon)</td>
<td>material implication or conditional</td>
</tr>
<tr>
<td>§xiîmata •pl≠</td>
<td>•pl≠</td>
</tr>
</tbody>
</table>

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<td>•pl≠</td>
</tr>
</tbody>
</table>

criteriology and much psychology of knowledge”, I. M. BOCHENSKI, Ancient Formal Logic, Amsterdam, 1951, p. 83.


4 Note that the Stoics knew various forms of implication like Diodorean and Connective besides the Philonian one. Cf. I. M. BOCHENSKI, Ancient Formal Logic, cit. pp. 89-90.
(simple propositions) | (conditional) | \( p \supset q \)  
--- | --- | ---  
\( \text{lhqû$} \) (true) | \( \text{lhqû$} \) (true) | 1  
\( \text{lhqû$} \) (false) | \( \text{ye,do$} \) (false) | 0  
\( \text{ye,do$} \) (true) | \( \text{lhqû$} \) (true) | 1  
\( \text{ye,do$} \) (false) | \( \text{lhqû$} \) (true) | 1  
\( \text{ye,do$} \) (true) | \( \text{lhqû$} \) (true) | 1  
\( \text{ye,do$} \) (true) | \( \text{lhqû$} \) (true) | 1  
\( \text{ye,do$} \) (false) | \( \text{ye,do$} \) (false) | 0  
\( \text{ye,do$} \) (false) | \( \text{ye,do$} \) (false) | 0  

See the following table for a comparison of the second logical function:

<table>
<thead>
<tr>
<th>STOICS (diezeugmûnon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MODERN LOGIC exclusive disjunction or non-equivalence</td>
</tr>
<tr>
<td>(simple propositions)</td>
</tr>
<tr>
<td>( \text{xlímata •pl#} )</td>
</tr>
</tbody>
</table>
| \( \text{lhqû$} \) (true) | \( \text{lhqû$} \) (true) | 1 | 1 | 0  
| \( \text{lhqû$} \) (false) | \( \text{ye,do$} \) (false) | 1 | 0 | 1  
| \( \text{ye,do$} \) (true) | \( \text{lhqû$} \) (true) | 0 | 1 | 1  
| \( \text{ye,do$} \) (false) | \( \text{ye,do$} \) (false) | 0 | 0 | 0  

See the following table for a comparison of the third logical function:

<table>
<thead>
<tr>
<th>STOICS (πεπδiezeugmûnon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MODERN LOGIC inclusive disjunction or logical sum</td>
</tr>
<tr>
<td>(simple propositions)</td>
</tr>
<tr>
<td>( \text{xlímata •pl#} )</td>
</tr>
</tbody>
</table>
| \( \text{lhqû$} \) (true) | \( \text{lhqû$} \) (true) | 1 | 1 | 1  
| \( \text{lhqû$} \) (false) | \( \text{ye,do$} \) (false) | 1 | 0 | 1  
| \( \text{ye,do$} \) (true) | \( \text{lhqû$} \) (true) | 0 | 1 | 1  
| \( \text{ye,do$} \) (false) | \( \text{ye,do$} \) (false) | 0 | 0 | 0  

3
3. Galen’s logic: dyadic logical functions

Galen makes indiscriminate use in of the Aristotelian terminology protaseis (pro\(\text{t}\)\(\text{a}\)s\(\text{e}\)s) and the Stoic one axiomata (\(\xi\)\(\text{l}\)\(\text{m}\)a\(\text{t}\)a) for propositions. He tell us in several places that the words «to be,» «to exist,» «to happen,» and «to be true,» all signify the same thing. These words seem to have for him the significance of the Stoic «fact» or «happening.».\(^5\) Galen speaks of categorical and hypothetical instead of simple and non-simple propositions.\(^6\) See the following table:

<table>
<thead>
<tr>
<th>STOICS</th>
<th>GALEN</th>
<th>MODERN LOGIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\xi)(\text{l})(\text{m})a(\text{t})a</td>
<td>(\text{(\phi)(\text{h})}) or (\text{(\xi)(\text{l})(\text{m})a(\text{t})a})</td>
<td>kathgorik(\text{a}) or (\xi)(\text{l})(\text{m})a(\text{t})a</td>
</tr>
<tr>
<td>(\text{(\omega)(c)(\phi)(\text{h})})</td>
<td>(\text{(\phi)(\text{t})(\text{m})(\text{a})s})</td>
<td>(\text{(\xi)(\text{l})(\text{m})a(\text{t})a})</td>
</tr>
</tbody>
</table>

\(\text{\(\xi\)\(\text{l}\)\(\text{m}\)a\(\text{t}\)a}\) or \(\xi\)\(\text{l}\)\(\text{m}\)a\(\text{t}\)a |

Institutio logica by Galen includes the first rigorous analysis of the connective ‘\(\text{\(\tau\)\(o\)i}\)’ or ‘\(\text{\(i\)}\)’. Pergamus’ physician distinguishes the case in which there are only two categorical propositions and that in which categorical propositions are more than two. In the first place he speaks of a complete battle or conflict (m\(\text{s}\)\(\text{c}\)\(\text{h}\) te\(\text{l}\)\(\text{a}\)\(\text{a}\)) between two categorical propositions and an incomplete battle or conflict (m\(\text{s}\)\(\text{c}\)\(\text{h}\) ù\(\text{l}\)\(\text{i}\)\(\text{l}\)\(\text{p}\)\(\text{o}\)\(\text{j}\)) always between two categorical propositions. The first logical function involves the reality of one and only one state of affairs corresponding to such categorical propositions, the second considers also the eventuality


that no state of things corresponds to such categorical propositions. It is clear that the complete battle (mých telea) between two categorical propositions described by Galen corresponds to the Stoic disjunction (diezeugmînon), and therefore to the modern exclusive disjunction or non-equivalence (‘J’ in Łukasiewiczian symbolism and ‘≠’ in the standard one). You will note that whereas the Stoics use metalanguage concerning propositions (i.e “true” and “false”) Galen, in his capacity of medical scientist and physician, prefers to refer to the existence or the non-existence of states of affairs, that establish the truth or falsity of propositions. See the following table:

<table>
<thead>
<tr>
<th><strong>STOICS</strong></th>
<th><strong>GALEN</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>(diezeugmînon)</td>
<td>(mých telea)</td>
</tr>
<tr>
<td>(disjunction)</td>
<td>(complete battle)</td>
</tr>
<tr>
<td><strong>¶fimata ¶f#</strong></td>
<td><strong>prot™sei$ kathgorikaà</strong></td>
</tr>
<tr>
<td>(simple propositions)</td>
<td>(categorical propositions)</td>
</tr>
<tr>
<td><strong>¶lhqû$</strong></td>
<td><strong>m¬ eênai</strong></td>
</tr>
<tr>
<td>(true)</td>
<td>(to be)</td>
</tr>
<tr>
<td><strong>¶lhqû$</strong></td>
<td><strong>m¬ eênai</strong></td>
</tr>
<tr>
<td>(true)</td>
<td>(not to be)</td>
</tr>
<tr>
<td><strong>ye, do$</strong></td>
<td><strong>eênai</strong></td>
</tr>
<tr>
<td>(false)</td>
<td>(to be)</td>
</tr>
<tr>
<td><strong>ye, do$</strong></td>
<td><strong>m¬ eênai</strong></td>
</tr>
<tr>
<td>(false)</td>
<td>(not to be)</td>
</tr>
<tr>
<td><strong>ye, do$</strong></td>
<td><strong>eênai</strong></td>
</tr>
<tr>
<td>(false)</td>
<td>(false)</td>
</tr>
<tr>
<td><strong>ye, do$</strong></td>
<td><strong>m¬ eênai</strong></td>
</tr>
<tr>
<td>(false)</td>
<td>(false)</td>
</tr>
</tbody>
</table>

It is clear also that the incomplete battle (mách üllipoj) between two categorical propositions considered by Galen corresponds to the non-conjunction of modern logic, the ‘nand’ function of computer science (‘D’ in Łukasiewiczian symbolism and ‘1’ in the standard one). See the following table:

<table>
<thead>
<tr>
<th><strong>GALEN</strong></th>
<th><strong>MODERN LOGIC</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>(mách üllipoj)</td>
<td>non-conjunction or incompatibility</td>
</tr>
<tr>
<td>incomplete battle</td>
<td></td>
</tr>
<tr>
<td>prot™sei$ kathgorikaà</td>
<td>m¬ eênai</td>
</tr>
<tr>
<td>(categorical propositions)</td>
<td>(incomplete battle)</td>
</tr>
<tr>
<td><strong>eênai</strong></td>
<td><strong>m¬ eênai</strong></td>
</tr>
<tr>
<td><strong>eênai</strong></td>
<td><strong>m¬ eênai</strong></td>
</tr>
<tr>
<td><strong>m¬ eênai</strong></td>
<td><strong>m¬ eênai</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>pq</strong></th>
<th><strong>Dpq or</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1</td>
<td>0</td>
</tr>
</tbody>
</table>
4. Galen’s logic: polyadic logical functions

The case that conflicting categorical propositions are more than two has greater significance. Galen writes:

«But now let us assign their names to these propositions: so then, for clear and concise exposition, nothing prevents our calling propositions with complete conflict “disjunctives” (diezeugmûna), and those of incomplete conflict “quasi-disjunctives” (parapløsia diezeugmûnoij); let there be no quibble over whether to say “quasi” (parapløsia) or “like-disjunctives” (÷moia); but in some propositions it is possible for more than one or for all the members to be true, and necessary for one to be true; some call the propositions of this sort “paradisjunctives” (paradiezeugmûna), since the disjunctives have one member only true, whether they be composed of two simple propositions or of more than two. For “Dion is walking” (Dàwn peripateé) is one simple proposition, and so also “Dion is sitting” (kßqhtai Dàwn); and “Dion is lying down” (katßkeitai Dàwn) is one proposition, and so, too, “He is running” (trûcei), and “He is standing still” (üsthken), but out of all of them is made a disjunctive proposition, as follows: “Dion either is walking or is sitting or is lying down or is running or is standing still” (Dàwn ¡toi peripateé f kßqhtai f katßkeitai f trûcei f üsthken); whenever a proposition is composed in this way any one member is in incomplete conflict with each of the other members, but taken all together they are in complete conflict with one another, since it is necessary that one of them must be true and the others not.» Gal. Inst. log. V, 1-2.

In the 1940s James W. Stakelum traces the truth table of diezeugmûnon with three simple propositions by means of informal language starting from Galen’s text. See the following table:

<table>
<thead>
<tr>
<th>(to be)</th>
<th>(to be)</th>
<th>(not to be)</th>
<th>1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>eênai</td>
<td>m¬ eênai</td>
<td>eênai</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(to be)</td>
<td>(not to be)</td>
<td>(to be)</td>
<td>eênai</td>
<td>eênai</td>
<td>eênai</td>
</tr>
<tr>
<td>(not to be)</td>
<td>m¬ eênai</td>
<td>m¬ eênai</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

If the first member is and the second is and the third is the excluding alternative proposition is
Stakelum observes rightly: “Necessarily one possibility must be true. There is also the added consideration that, according to the nature of the possibilities, only one can be true at a given time. Of all the members considered together, one must be true; therefore, there is among them complete opposition; which signifies that one member must be true and the others false. But of any two members of the group, though they both cannot be true together, it is not necessary that one be true, for so one of the other members can be true; therefore, between any two members there is only incomplete opposition, that is, they both can possibly be not true”.7

Let ‘$p_1$’, ‘$p_2$’, ‘$p_3$’ be simple or categorical propositions. Let us symbolise the above truth table.

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>diezeugmínon (disjunction)</th>
<th>mach teleàa (complete battle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A.1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(A.2)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(A.3)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(A.4)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(A.5)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(A.6)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(A.7)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(A.8)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

It is clear that as far as the complete battle between ‘$p_1$’,

7 W. STAKELOM, *Galen and the Logic of Proposition*, cit., p. 32.
‘$p_1$’ and ‘$p_2$’ is concerned, the cases (A.4), (A.6), (A.7) hold; as far as the incomplete battle between ‘$p_1$’ and ‘$p_2$’ is concerned only the cases (A.4) and (A.6) hold; as far as the incomplete battle between ‘$p_2$’ and ‘$p_3$’ is concerned only the cases (A.6) and (A.7) hold. In other words when the conflicting simple propositions are three, one complete and three incomplete battles take place.

Stakelum does not outline the truth table of paraplosion $t$, $\text{diezeugmûn}_3$ with three simple propositions, but which concerns a $m$-adic language universe ($m>3$), nevertheless he traces the paradiezeugmûnon one.\(^8\)

<table>
<thead>
<tr>
<th>If the first member is</th>
<th>and the second is</th>
<th>and the third is</th>
<th>the not-excluding alternative proposition is</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>True</td>
<td>True</td>
<td>False</td>
<td>True</td>
</tr>
<tr>
<td>True</td>
<td>False</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>True</td>
<td>False</td>
<td>False</td>
<td>True</td>
</tr>
<tr>
<td>False</td>
<td>True</td>
<td>True</td>
<td>True</td>
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<tr>
<td>False</td>
<td>True</td>
<td>False</td>
<td>True</td>
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<tr>
<td>False</td>
<td>False</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>False</td>
<td>False</td>
<td>False</td>
<td>False</td>
</tr>
</tbody>
</table>

Symbolising the above Stakelum’s truth table, we obtain the following:

\(^8\)“This is one of the clearest explanations found in Galen’s Introduction. There is not the slightest doubt about the nature of not-excluding alternative as contrasted with the excluding alternative. The not-excluding alternative is a proposition of two or more members, of which one must be true but of which more than one or all may be true. This use of «or» in this proposition is similar to the Latin «vel».”, J. W. STAKELUM, Galen and the Logic of Proposition, cit., pp. 35-36.
You can observe that if we have to do with the paradiezeugmûnon composed of three categorical propositions, then cases from (A.1) to (A.7) hold. (A.8) is the only case that does not hold.

But at this point a very great difficulty arises. You can verify that whereas Galen’s \( n \)-adic paradiezeugmûnon \( (n\geq 3) \) coincides with the polyadic extension of logical sum, neither the \( n \)-adic diezeugmûnon \( (n\geq 3) \) coincides with the extension of non-equivalence, nor the \( n \)-adic parapløsion t, diezeugmûnJ \( (n\geq 3) \) – but concerning a speech universe of \( m \)-adic simple propositions \( (m>3) \) – coincides with the extension of non-conjunction (the “nand” interpretation of Sheffer’s functor). Jürgen Mau has observed rightly: “Hier liegen logistisch gesehen zwei \( n \)-adische Wahrheitswertfunktoren vor, wir können für die symbolische Darstellung nicht die dyadischen Funktoren, die Contravalenz (\( J \)) und den Shefferschen Funktor (\( D \)) verwenden. Es wird ein komplexer Ausdruck mit verschiedenen dyadischen Funktoren benötigt”.\(^9\) The reason is clear. The modern concept of exclusive disjunction between atomic \( n \)-sentences is not conceived like a simultaneous battle or conflict between \( n \) competitors \( (n\geq 3) \) but always like a battle between two rivals – let us imagine a boxing

\(^9\) GALEN, *Einführung in die Logik*, Kritisch-exegetischer Kommentar mit deutscher Übersetzung von J. Mau, Berlin, 1960, p. 16. For (a) the complete calculus of the 3-argumental extension of non-equivalence, which falsifies the hypothesis that such an extension coincides with diezeugmûnon, and (b) the complete calculus of the 3-argumental extension of non-conjunction, which falsifies the hypothesis that such an extension coincides with parapløsion t, diezeugmûn, see M. MALATESTA, «Classical Fundamentals and a Modern Foundation of the Probability Calculus», in *Metalogicon* II (1989), pp. 94-132.
match –. If the rivals are more than two, then the winner enters the lists with a new challenger, and so on, until one and only one is the final winner and the others losers. In the ancient world the concept is quite different. You must imagine one ring inside which several boxers fight to the last simultaneously, until one and only one is the final winner and the others losers. Analogous considerations can be made as far as the incomplete battle is concerned, where the adversaries struggle until either one and only one is the final winner and the others losers, or all are losers.

We have to give credit to Mau for the first rigorous definitions of complete and incomplete battles.

As far as the complete battle is concerned the German scholar gives the following definition in Polish symbolism

$$KKKAapqrCpKNqNrCqKNpNrCqKNpNq.$$ 

Now, if we substitute ‘$p_1$’ for ‘$p’’, ‘$p_2$’ for ‘$q’ and ‘$p_3$’ for ‘$r’ and use Hilbert’s symbolism, then the above definition becomes

$$(p_1 \lor p_2 \lor p_3) \& (p_1 \rightarrow \overline{p_1} \& \overline{p_2}) \& (p_2 \rightarrow \overline{p_1} \& \overline{p_2}) \& (p_3 \rightarrow \overline{p_1} \& \overline{p_2})$$

The sense is obvious: at least one of the three propositions holds, but under this condition, – i.e. if the first holds, then the second and the third do not, and, if the second holds, then the first and the third do not, and, if the third holds, then the first and the second do not –.

The German scholar gives the following definition for the incomplete battle with three simple propositions, concerning therefore a speech universe of $m$ simple propositions ($m>3$):

$$KKCpKNqNrCqKNpNrCrKNpNq$$

i.e.

$$(p_1 \rightarrow \overline{p_1} \& \overline{p_2}) \& (p_2 \rightarrow \overline{p_1} \& \overline{p_3}) \& (p_3 \rightarrow \overline{p_1} \& \overline{p_3})$$

As you can see, this definition is a proper part of the first one.

A definition of paradiezeugmûnon with three simple propositions is superfluous since such a function is an extension of
the dyadic logical sum.\textsuperscript{10}

At this point we can outline a comparative truth table of Galen’s \textit{diezeugmûnon}, \textit{paraplosion $t$}, \textit{diezeugmûnJ} and \textit{paradiezeugmûnon} with three simple propositions:

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>diezeugmûnon (disjunction)</th>
<th>paraplosion $t$, diezeugmûnJ (quasi-disjunction)</th>
<th>para-diezeugmûnon (para-disjunction)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A.1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(A.2)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(A.3)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(A.4)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(A.5)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(A.6)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(A.7)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(A.8)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Definitions more useful for a logical calculus, given their recursive character, can be formulated in terms of disjunctive normal forms, which are, as well known, logical sums of elementary logical products.

For \textit{diezeugmûnon} or \textit{mîch teleàa} with three simple propositions we have the following definition:

\[(p_1 \& \overline{p}_2 \& \overline{p}_3) \lor (\overline{p}_1 \& p_2 \& \overline{p}_3) \lor (\overline{p}_1 \& \overline{p}_2 \& p_3)\].

The meaning is evident: either the first proposition holds but the second and the third do not, or the first proposition does not hold but the second does and the third does not, or the first and the second propositions do not hold but the third does.

For \textit{paraplosion $t$}, \textit{diezeugmûnJ} with three simple propositions but concerning a speak universe with $m$ simple propositions ($m>3$), we have the following definition:

\[(p_1 \& \overline{p}_2 \& \overline{p}_3) \lor (\overline{p}_1 \& p_2 \& \overline{p}_3) \lor (\overline{p}_1 \& \overline{p}_2 \& p_3) \lor (\overline{p}_1 \& \overline{p}_2 \& \overline{p}_3)\].

Also the sense of this definition is clear: either the first proposition holds but the second and the third do not, or the first proposition does not hold but the second does and the third does not, or the first and the second propositions do not hold but the third does, or none of these three propositions holds.

Now, if we consider that with \( n \) simple propositions exactly \( 2^n \) possible worlds representable as elementary logical products are generated, then with three simple propositions we obtain eight possible worlds. See the following comparative table:

<table>
<thead>
<tr>
<th>Elementary logical products</th>
<th>diezeugmûnon (disjunction)</th>
<th>paraplosion t, diezeugmûn (quasi-disjunction)</th>
<th>para-diezeugmûnon (para-disjunction)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>( p &amp; q &amp; r )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>( p &amp; q &amp; \overline{r} )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( w_3 )</td>
<td>( p &amp; \overline{q} &amp; r )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( w_4 )</td>
<td>( \overline{p} &amp; q &amp; r )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( w_5 )</td>
<td>( \overline{p} &amp; q &amp; \overline{r} )</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

You can observe the following:

– in the case of diezeugmûnon with three simple propositions the logical sum of worlds \( w_4, w_6, \) and \( w_7 \) hold;

– in the case of paraplosion t, diezeugmûnJ with three simple propositions, but concerning a speak universe of \( m \) simple propositions (\( m>3 \)), the logical sum of worlds \( w_4, w_6, w_7, \) and \( w_8 \) hold;

– in the case of paradiezeugmûnon with three simple propositions, the logical sum of worlds from \( w_4 \) to \( w_7 \) hold. Only the world \( w_8 \) does not.

5. Extension of Stoic logic. Extension of the first two Chrysippus’ undemonstrateds
Chrysippus speaks of five basic types of undemonstrated arguments. They are called “undemonstrated” because they have no need of demonstration owing to their being immediately obvious (mê crean †contej †podeâxewj t, âxt’qen eênai perifanûj).

The first undemonstrated is that which, from a conditional and its antecedent, deduces the consequent as a conclusion.

«Such then – to put it briefly – is the construction of the hypothetical proposition, and a proposition of this kind seems to promise that its second logically follows its first, and that if the antecedent exists the consequent will exist (Ω m°n oÂn sostasij to, suñhmûnou, Éj ūn sunt’md eâipen, ûstî toiaîth, ûpaggûlesqai de dokêê tî tîoî, tîon axâsma ñkolouqêîn t, ūn âît, prîl tî ūn âît, deîteron kaî õntoj to, Ùgoumûnou Ùsesqai tî Ùgon)». Sext. Emp. Adv. Math. VIII, 111.

We read the following example:

«If the first, the second; but the first; therefore the second (eá tî prÔton, tîy deîteron: tîy ðû ge prÔton: tîy ñra deîteron)», Sext. Emp. Adv. Math. VIII, 227.

In modern symbols:

\[
\begin{align*}
p_1 \rightarrow p_2 \\
p_1 \quad \vdash p_2
\end{align*}
\]

The second undemonstrated is the argument which, from a conditional and the contradictory of its consequent, deduces the contradictory of its antecedent.

We read the following example:

\[11 \text{ SEXT. EMP. } \text{Adv. Math. VIII, 223.} \]
\[12 \text{ SEXT. EMP. } \text{Adv. Math. VIII, 224.} \]
\[13 \text{ We found a like formulation in Diogenes Laertius } \text{“If the 1st, the 2nd too; but certainly the first; therefore the second”. DIOG. LAERT. } \text{Vitae, VII, 39.} \]
\[14 \text{ SEXT. EMP. } \text{Adv. Math. VIII, 225.} \]
«If the first, the second; but not the second; therefore not the first (ἐά τύ πρῶτον, τύ δεύτερον: oὐκ εὖ δεύτερον: οὐκ ἄρα τύ πρῶτον).» (Sext. Emp. Adv. Math. VIII, 227)

In modern symbols we have:

\[
p_1 \rightarrow p_2
\]

\[
\bar{p}_2
\]

In order to extend both undemonstrated syllogisms, Galen considers the following example:

«If nourishment is distributed from the belly to the whole body, either it undergoes this through self-motion (τοι ἄν αὐτῷ ἀναζωτεῖ τὴν ἐκπαράστασιν) or by being sent by the stomach (φυπὸ τῶν γαστρῶν πεπομένη), or by being attracted by the parts (φυπὸ τῶν μοραών ὑλομένη), or conducted by the veins (φυπὸ τῶν φλεβῶν παραγόμενη)». (Gal. Inst. log., XV, 7.)

Then the Pergamus’ physicist, after having criticised the lack of understanding of such an inference by some people, adds:

«But this (sc. inference) has the same force as the first undemonstrated syllogism of the hypotheticals, the distribution of nourishment being by hypothesis antecedent, and what was said next following as consequent, and it makes no difference whether the inferred conclusion is, in respect to its material, disjunctive (diezumένη) or paradisjunctive (paradiezugmένη); for the force of the first undemonstrated is consistent with either of the forms, since it is as follows: ‘if the first, either the second or the third or the fourth or the fifth’; then the minor premise, ‘but the 1st; therefore, either the 2nd or the 3rd or the 4th or the 5th’; a second minor premise, in the manner of the second undemonstrated is the following: ‘but neither the 2nd or the 3rd nor the 4th or the 5th; therefore not the 1st’.» (Gal. Inst. log., XV, 8)\(^\text{15}\)

\(^{15}\) Note that Galen applies the Ockham-De Morgan rules. In fact the second premise of the second undemonstrated “neither the 2nd or the 3rd nor the 4th or the 5th”, composed by a binary rejection of two logical sums of simple
Galen analyses together the extension of the first and second undemonstrated. In extending the first undemonstrated he considers two cases: the first, when the consequent is a polyadic disjunctive, and the second, when it is a polyadic paradisjunctive. In extending the second undemonstrated he does not make such a distinction but we are allowed to think that it holds in such an extension too. Therefore the extension of the first undemonstrated has two schemes of inference which we shall symbolise and label by marks I.1 and I.2:

\[
\begin{align*}
p_1 &\rightarrow (p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \lor \\
& \lor (\bar{p}_2 \& p_1 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_3 \& p_1 \& p_2 \& \bar{p}_2 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_4 \& p_1 \& p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_5 \& p_1 \& p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4) \\
& \\
& \frac{p_1}{(p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \lor} \\
& \lor (\bar{p}_2 \& p_1 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_3 \& p_1 \& p_2 \& \bar{p}_2 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_4 \& p_1 \& p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_5 \& p_1 \& p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4) \\
& \\
& I.1 \\

p_1 \rightarrow p_2 \lor p_3 \lor p_4 \lor p_5 \\
\frac{p_1}{p_2 \lor p_3 \lor p_4 \lor p_5} \\
I.2
\end{align*}
\]

Also the extension of the second undemonstrated has two schemes of inference, labelled by marks II.1 and II.2:

\[
\begin{align*}
p_1 &\rightarrow (p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \lor \\
& \lor (\bar{p}_2 \& p_1 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_3 \& p_1 \& p_2 \& \bar{p}_2 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_4 \& p_1 \& p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_5 \& p_1 \& p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4) \\
& \\
& \frac{p_1}{(p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \lor} \\
& \lor (\bar{p}_2 \& p_1 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_3 \& p_1 \& p_2 \& \bar{p}_2 \& p_4 \& \bar{p}_4 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_4 \& p_1 \& p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_5 \& \bar{p}_5) \\
& \lor (\bar{p}_5 \& p_1 \& p_2 \& \bar{p}_2 \& p_3 \& \bar{p}_3 \& p_4 \& \bar{p}_4) \\
& \\
& \end{align*}
\]

propositions, is equivalent to a logical product of four negated simple propositions.
6. Extension of the last three Chrysippus’ undemonstra-teds

According to Chrysippus the third undemonstrated is the argument which, for its first premise has the negation of a conjunction; for its second premise, one of the conjuncts; for its conclusion the negation of the other conjunct. Therefore we have two inference schemes. The first is the following:

\[
\begin{align*}
& p_1 \land p_2 \\
& \quad p_1 \\
& \quad p_2 \\
& \quad \overline{p}_1 \\
\end{align*}
\]

The second is the following:

\[
\begin{align*}
& p_1 \land p_2 \\
& \quad p_2 \\
& \quad \overline{p}_1 \\
\end{align*}
\]

\[\text{SEXT. EMP. Adv. Math. VIII, 226.}\]
The fourth undemonstrated is an argument which, for its first premise has a disjunction; for its second premise one of the members of the disjunction; for its conclusion the negation of the other member.\textsuperscript{17} Therefore we have two inference schemes. The first is the following:

\[
\begin{align*}
\neg p_1 & \neq p_2 \\
\hline
\neg p_1 & \\
\hline
p_2 &
\end{align*}
\]

The second is the following:

\[
\begin{align*}
\neg p_1 & \neq p_2 \\
\hline
p_2 & \\
\hline
\neg p_1 &
\end{align*}
\]

The fifth undemonstrated is an argument which, for its first premise has a disjunction; for its second premise the negation of one of the members of the disjunction; for its conclusion the other member.\textsuperscript{18} Therefore we have two inference schemes. The first is the following:

\[
\begin{align*}
\neg p_1 & \neq p_2 \\
\hline
\neg p_1 & \\
\hline
p_2 &
\end{align*}
\]

The second is the following:

\[
\begin{align*}
\neg p_1 & \neq p_2 \\
\hline
p_2 & \\
\hline
\neg p_1 &
\end{align*}
\]

\textsuperscript{17} SEXT. EMP. \textit{Pyrr. Hyp.} II, 158.

\textsuperscript{18} SEXT. EMP. \textit{Pyrr. Hyp.} II, 158.
Galen speaks of the extension of the above undemonstrated syllogisms in several passages of his book of logic. In chapter V the Pergamus’ physician writes:

«In the case of complete battle two syllogisms can be constructed if we take as an additional premise that one of the members is true, or on the contrary is not true, and infer that the second is not true when the first one is, or is true when the first is not; but for incomplete battle there is but one additional premise, that one member is true, and but one conclusion, that the remaining member is not true. It is this way when the battle consists of two members; when there are more than two conflicting members, in the case of complete battle, we can, either asserting one member to be true, deny all the rest, or denying all the rest, assert the one member to be true; it is not possible, however, by denying the one, to allow the rest be true, or, asserting the rest, to deny that the one is true; on the other hand, in the case of incomplete battle, by asserting some one member, we can deny the remaining number, but we will have no other additional premise suitable for producing a syllogism». Gal. Inst. log., V, 3-4.

In this passage the Greco-Roman physician has a clear awareness of his new viewpoint that overtakes the Stoic perspective, which, in this way, becomes the extreme inferior limit of a potentially infinitary logic. At the end of chapter XIV Galen observes too:

«There will be two syllogisms deriving from complete consequence, and another two from complete battle, and let those from consequence be called first and second, and those from battle, fourth and fifth, since Chrysippus put it so; but the third, in expression the same as Chrysippus’s, but according to the nature of the things postulated, is not the same; for its genesis is not, as he thought, from a negative conjunctive, but from deficient battle, and it has one affirmative additional assumption, not two, as does either of those kinds derived from complete consequence and complete battle». (Gal. Inst. log., XIV, 11)

The above passage is very important. In fact whereas in the case of two conflicting simple propositions, whether Chrysippus’s definition of the major premise of third undemonstrated or that of
Galen holds, because both are the extreme inferior limits of two
different polyadic functions, i.e. the polyadic “nand” in the first
case and the polyadic incomplete battle in the second, when you
extend the third undemonstrated to \( n \) simple sentences, \((n \geq 3)\) but
concerning a speech universe of \( m \) simple sentences \((m > 3)\), such
an extension holds only on the base of Galen’s definition.

By translating Galen’s informal speech into the formal
one, we obtain for the third undemonstrable with three simple
propositions the following inference schemes, that we shall call
III.1, III.2, III.3:

\[
\begin{align*}
(p_i & \land \overline{p_i} & \land \overline{p_i}) v \\
\lor (\overline{p_i} & \land p_i & \land \overline{p_i}) v \\
\lor (\overline{p_i} & \land \overline{p_i} & \land p_i) v \\
\lor (p_i & \land \overline{p_i} & \land \overline{p_i}) v \\
\hline
p_i, \\
\overline{p_i} & \land \overline{p_i}
\end{align*}
\]

III.1

\[
\begin{align*}
(p_i & \land \overline{p_i} & \land \overline{p_i}) v \\
\lor (\overline{p_i} & \land p_i & \land \overline{p_i}) v \\
\lor (\overline{p_i} & \land \overline{p_i} & \land p_i) v \\
\lor (p_i & \land \overline{p_i} & \land \overline{p_i}) v \\
\hline
p_i, \\
\overline{p_i} & \land \overline{p_i}
\end{align*}
\]

III.2

\[
\begin{align*}
(p_i & \land \overline{p_i} & \land \overline{p_i}) v \\
\lor (\overline{p_i} & \land p_i & \land \overline{p_i}) v \\
\lor (\overline{p_i} & \land \overline{p_i} & \land p_i) v \\
\lor (p_i & \land \overline{p_i} & \land \overline{p_i}) v \\
\hline
p_i, \\
\overline{p_i} & \land \overline{p_i}
\end{align*}
\]

III.3
By translating Galen’s informal speech into the formal one, we obtain for the fourth undemonstrable with three simple propositions the following inference schemes, that we shall call IV.1, IV.2, IV.3:

\[
\begin{align*}
   & (p_1 \land p_2 \land p_3) \lor \\
   & (\neg p_1 \land p_2 \land \neg p_3) \lor \\
   & (p_1 \land \neg p_2 \land p_3) \\
   \hline
   & p_1 \\
   \end{align*}
\]

IV.1

\[
\begin{align*}
   & (p_1 \land p_2 \land p_3) \lor \\
   & (\neg p_1 \land p_2 \land \neg p_3) \lor \\
   & (\neg p_1 \land \neg p_2 \land p_3) \\
   \hline
   & p_2 \\
   \end{align*}
\]

IV.2

\[
\begin{align*}
   & (p_1 \land p_2 \land p_3) \lor \\
   & (\neg p_1 \land p_2 \land \neg p_3) \lor \\
   & (\neg p_1 \land \neg p_2 \land p_3) \\
   \hline
   & p_3 \\
   \end{align*}
\]

IV.3

By translating Galen’s informal speech into the formal one, we obtain for the fifth undemonstrable with three simple propositions the following inference schemes, that we shall call V.1, V.2, V.3:

\[
(p_1 \land \neg p_2 \land \neg p_3) \lor
\]
7. *Reductio ad unum* of Galen’s extension of Chrysippus’s undemonstrated syllogisms

Let us define the complete battle as follows:

\[
\begin{align*}
\hat{M}^{\forall}_{J^p,} p_i &= \forall (p_i \land \bar{p}_i) \lor (\bar{p}_i \land p_i) \\
\hat{M}^{\forall}_{J^r,} p_i &= \forall (p_i \land \bar{p}_i \land \bar{p}_j) \lor (\bar{p}_i \land p_i \land \bar{p}_j) \lor (\bar{p}_i \land \bar{p}_j \land p_i)
\end{align*}
\]

In general for \( n \) conflicting simple propositions (\( n \geq 3 \))
Let us define the incomplete battle of two conflicting simple propositions, but concerning a speech universe of \( m \) conflicting simple propositions \((m>2)\), as follows:

\[
M^X\!\downarrow \ p_i = \phi (p_i \& \overline{p}_i \& \ldots \& \overline{p}_m) \lor (\overline{p}_i \& p_{i+1} \& \ldots \& \overline{p}_m) \lor \ldots \\
\ldots \lor (\overline{p}_i \& \overline{p}_{i+1} \& \ldots \& \overline{p}_m)
\]

Let us define the incomplete battle of three conflicting simple propositions, but concerning a speech universe of \( m \) conflicting simple propositions \((m>3)\), as follows:

\[
M^X\!\downarrow \ p_i = \phi (p_i \& \overline{p}_i \& \overline{p}_{i+1} \lor (\overline{p}_i \& p_{i+1} \& \overline{p}_{i+2}) \lor \ldots \\
\ldots \lor (\overline{p}_i \& \overline{p}_{i+1} \& \overline{p}_{i+2}) \lor (\overline{p}_i \& p_{i+2} \& \overline{p}_{i+3}) \lor \ldots \\
\ldots \lor (\overline{p}_i \& \overline{p}_{i+1} \& \overline{p}_{i+2} \& \ldots \& \overline{p}_m)
\]

In general for \( n \) conflicting simple propositions \((n>3)\), but concerning a speech universe of \( m \) conflicting simple propositions \((m>n)\), we have the following definition:

\[
M^X\!\downarrow \ p_i = \phi (p_i \& \overline{p}_i \& \ldots \& \overline{p}_m) \lor (\overline{p}_i \& p_{i+1} \& \ldots \& \overline{p}_m) \lor \ldots \\
\ldots \lor (\overline{p}_i \& \overline{p}_{i+1} \& \ldots \& \overline{p}_m)
\]

Let the following definitions for paradisjunction be:

\[
A^2\!\downarrow \ p_i = \phi p_i \lor p_{i+1}
\]
\[
A^3\!\downarrow \ p_i = \phi p_i \lor p_{i+1} \lor p_{i+2}
\]

\[
\ldots \ldots
\]

In general for \( n \) simple propositions \((n>3)\) we have:

\[
A^n\!\downarrow \ p_i = \phi p_i \lor p_{i+1} \lor \ldots \lor p_n
\]
Now if we adopt Łukasiewicz’s symbol ‘\(N\)’ for negation and interpret the metalinguistic sign ‘\(-\)’ as “minus” or “excepted”, we can translate and generalise the above inference schemes as follows:

\[
\begin{array}{ll}
\text{I.1} & Cp_i \rightarrow Mp_i \\
& \frac{p_i}{Mp_i} \\
\text{II.1} & Cp_i \rightarrow Ap_i \\
& \frac{p_i}{Ap_i} \\
\text{I.2} & Cp_i \rightarrow Np_i \\
& \frac{\rightarrow Np_i}{Mp_i} \\
\text{II.2} & Cp_i \rightarrow NAp_i \\
& \frac{\rightarrow NAp_i}{Np_i} \\
\end{array}
\]
8. Conclusion

The above inference schemes are clearly instruments of medical diagnosis.

E.g. if a symptom involves a set of mutually incompatible illnesses, and there is such a symptom, therefore we have to do with such a set of mutually incompatible illnesses (by I.1). At this point we ask: is such a set of mutually incompatible illnesses complete or incomplete? If it is incomplete and we have ascertained that there is one certain illness, then we can exclude the remaining others (by III.1.2.3); if it is complete, then not only is it possible to go on in the same way as in the former case (by IV.1.2.3), but, excluding progressively one illness after the other, we shall find the illness that is the case (by V.1.2.3).
Another instance: if an illness involves a set of consequences compatible among themselves and no such consequences take place, then not even such an illness will occur (by II.2). And so on.

References

a) Sources


b) Commentaries and Researches

Philosophy of Science, Sections 6-9, p. 144; extended text in “Metalogicon”, (1992) V, 2, pp. 73-102.
– STAKELUM J. W., Galen and the logic of proposition, Angelicum, Roma, 1940.

\[(p \land \overline{p} \land \overline{p}) \lor
\lor(p \land \overline{p} \land \overline{p}) \lor
\lor(p \land \overline{p} \land \overline{p}) \lor
\overline{p} \land \overline{p},\]

\[(p \land \overline{p} \land \overline{p}) \lor
\lor(p \land \overline{p} \land \overline{p}) \lor
\lor(p \land \overline{p} \land \overline{p}) \lor
\overline{p} \land \overline{p},\]